Operator Domains in Nonclassical Measurement Error: A Comment on Hu and Schennach (2008)

Chae-Yeon Xon
Yonsei University

September 2026

Abstract

Hu and Schennach (2008) identify a model with nonclassical measurement error through similarities of observable integral operators and latent multiplication operators. Their proof treats the similarities and spectral projections as bounded on the ambient L^1 space. A wrapped Gaussian model satisfies Assumptions 1–5 while the observable similarity and displayed projections are unbounded; the density columns also lie outside the inverse operator's domain. Conjugated latent-band projections extend to a bounded spectral measure exactly under a uniform band-stability inequality. Combined with outcome separation and dense range, full band stability makes the measurement operator bounded below, which conflicts with smoothing by bounded densities on a nonatomic support. Equipping each candidate range with its transported graph norm restores the operator algebra. Reconstruction can then proceed under common-resolution or cyclic-orbit conditions. The factorization conclusion under Assumptions 1–5 remains an identification question.

JEL Codes: C14, C26, C31

Keywords: measurement error, spectral identification, integral operators, operator domains, latent variables

Yonsei University. eunixon@yonsei.ac.kr. Early stage draft; please do not cite or circulate without author's written permission. All errors are my own.

1. Introduction

Hu and Schennach (2008) study a latent regressor X^* observed through a measurement X and an instrument Z, with an outcome Y whose conditional law varies with X^*. Conditional independence factors the observable densities into integral operators. After inverting the observable X-given-Z operator on its range, the paper obtains a family of observable operators similar to multiplication by the conditional outcome density. Eigenvalues recover the outcome law and eigenfunctions recover the measurement-density columns; a location functional labels the latent coordinate.

The inverse in this construction is generally unbounded. The similarity identity first holds on a dense range. The proof on page 211 then uses boundedness of the latent multiplication function to treat the observable similarity as bounded and applies a spectral decomposition whose projections are $K\mathbf{1}_A K^{-1}$, where K is the measurement operator. Injectivity gives an algebraic inverse on $\mathcal{R}(K)$; bounded extension and a domain containing the density columns require further properties.

The operator identity and its domains provide the starting point. A probability model satisfying the five printed assumptions then shows where the boundedness, projection, and eigenfunction-domain steps fail. The analysis characterizes bounded extension of latent-band projections and follows the implications of full outcome separation for an ambient spectral repair. Transported range norms suggest a different operator space, and two reconstruction certificates state conditions under which that route succeeds. These results concern the published proof. Identification of the probability factorization under Assumptions 1–5 alone remains open.

TABLE 1. Domain requirements in the spectral route

Operation	Required property	Result in the wrapped-Gaussian model
KM_yK^{-1} on observable L^1	A uniform bound on the dense algebraic domain	The norm ratio in (9) diverges
KI_AK^{-1} as a spectral projection	Uniform latent-band stability	The band constant in (10) is infinite
Measurement-density column as an eigenfunction	Membership in the inverse operator's domain	The column lies outside $\mathcal{R}(K)$
Comparison across candidate factorizations	One common invariant calculus or graph space	Propositions 2 and the cyclic-orbit certificate state sufficient conditions

2. The observable similarity and its domain

For a fixed outcome value y, let $L_{x|z}$ be the integral operator with kernel $f_{X|Z}(x \mid z)$ and let $L_{y;x|z}$ use the joint conditional density $f_{Y,X|Z}(y, x \mid z)$. Write

$$K = L_{x|x^*}, \qquad H = L_{x^*|z}, \qquad (M_y m)(x^*) = f_{Y|X^*}(y \mid x^*) m(x^*). \tag{1}$$

The conditional-independence restrictions give

$$L_{x|z} = KH, \qquad L_{y;x|z} = KM_y H. \tag{2}$$

On $D = \mathcal{R}(L_{x|z})$, injectivity yields

$$T_y = L_{y;x|z} L_{x|z}^{-1} = KM_y K^{-1}. \tag{3}$$

Equation (3) is an algebraic identity on D. Assumption 3 and Lemma 1 make D dense in the ambient observable L^1 space. Density permits at most one bounded extension; its existence requires an operator bound on D.

For a latent measurable set A, the multiplication projection $I_A g = \mathbf{1}_A g$ is bounded on latent L^1. Its conjugate is initially

$$P_A^0(Kg) = K(\mathbf{1}_A g), \qquad Kg \in \mathcal{R}(K). \tag{4}$$

The formulas used to recover a measurement-density column treat (4) as a bounded spectral projection on observable L^1 and treat the column $K\delta_{x^*}$ as an eigenfunction in the operator domain. Both properties can be tested inside the statistical model.

3. A smoothing model under Assumptions 1–5

Let $\psi(r) = 2 \arctan r \in (-\pi, \pi)$ and $\tau(t) = \tan(t/2)$. Draw T uniformly on $(-\pi, \pi)$ and draw mutually independent wrapped-Gaussian errors E_x, E_y, E_z, independent of T. With addition modulo 2π, define

$$X^* = \tau(T), \qquad X = \tau(T + E_x), \qquad Y = \tau(T + E_y), \qquad Z = \tau(T + E_z). \tag{5}$$

All variables have support $\mathbb{R}$. If q_x is the wrapped-Gaussian density in angular coordinates, then

$$f_{X|X^*}(x \mid u) = q_x\{\psi(x) - \psi(u) \bmod 2\pi\}\psi'(x), \qquad \psi'(x) = \frac{2}{1 + x^2}. \tag{6}$$

The corresponding formulas hold for Y and Z.

The densities are positive and bounded; the coordinate Jacobians are bounded; and conditional independence follows from the independent errors. In angular coordinates, K is convolution by q_x with Fourier multipliers

$$\kappa_n = \exp(-\sigma_x^2 n^2/2) > 0. \tag{7}$$

The Z-given-X operator has nonzero multipliers as well. Fourier uniqueness gives the two injectivity conditions in Assumption 3. Distinct latent locations produce distinct shifted outcome densities on a set of positive marginal Y-probability, giving Assumption 4.

For a density p on $\mathbb{R}$, define the known location functional

$$M[p] = \tan\left\{\frac{1}{2}\operatorname{Arg}_{(-\pi,\pi)} \int_{\mathbb{R}} e^{i\psi(x)} p(x)\, dx\right\}. \tag{8}$$

For the column in (6), the integral has argument $\psi(u)$, so $M[f_{X|X^*}(\cdot \mid u)] = u$. This verifies Assumption 5.

The coordinate transformation is an L^1 isometry, so it suffices to calculate on the circle. Let $e_n(t) = e^{int}$ and let $m_y(t) = f_{Y|X^*}(y \mid t)$. For $u_n = Ke_n = \kappa_n e_n$, the $n-1$ Fourier coefficient of $KM_y e_n$ gives

$$\frac{\|T_y u_n\|_1}{\|u_n\|_1} \geq |\widehat{m}_y(-1)| \frac{\kappa_{n-1}}{\kappa_n} = |\widehat{m}_y(-1)| \exp\left\{\frac{\sigma_x^2(2n-1)}{2}\right\} \longrightarrow \infty. \tag{9}$$

Every u_n belongs to the natural domain D: if H has multiplier $b_n > 0$, then $u_n = KH(b_n^{-1}e_n)$. Thus (9) rules out a bounded ambient extension from the domain in (3), even though m_y is bounded.

Choose a level set A of m_y whose indicator has a nonzero first Fourier coefficient. The same calculation with multiplier $\mathbf{1}_A$ makes the conjugate KI_AK^{-1} unbounded. Finally, a measurement-density column is a circular shift of q_x. An L^1 preimage under K would have Fourier coefficients of unit modulus, contrary to the Riemann–Lebesgue lemma. The column therefore lies outside $\mathcal{R}(K)$ and outside D. This model satisfies Assumptions 1–5 and falsifies the bounded-extension, bounded-projection, and density-column-domain steps used in the printed spectral argument.

4. Band stability and spectral projections

Let (S, Σ, μ) be the latent space, $(\mathcal{X}, \nu)$ the observable space, and $K : L^1(\mu) \to L^1(\nu)$ an injective positive mass-preserving operator with dense range. For a sub-σ-field $\mathcal{G} \subseteq \Sigma$, define

$$C_{K,\mathcal{G}} = \sup_{A\in\mathcal{G}} \sup_{g\neq 0} \frac{\|K(\mathbf{1}_A g)\|_1}{\|Kg\|_1}. \tag{10}$$

PROPOSITION 1 (Band-stability criterion). *The maps P_A^0 in* (4)*, $A \in \mathcal{G}$, extend to uniformly bounded projections on $L^1(\nu)$ that satisfy the Boolean identities and strong countable additivity exactly when $C_{K,\mathcal{G}} < \infty$.*

PROOF. If the constant is finite, each P_A^0 extends uniquely from the dense range with norm at most $C_{K,\mathcal{G}}$. Complements, intersections, and finite disjoint unions pass from the range to its closure. For disjoint A_j and $u = Kg$,

$$\left\| P_{\cup_j A_j} u - \sum_{j=1}^{N} P_{A_j} u \right\|_1 = \left\| K\left(\mathbf{1}_{\cup_{j>N} A_j} g\right) \right\|_1 \le \left\| \mathbf{1}_{\cup_{j>N} A_j} g \right\|_1 \longrightarrow 0. \tag{11}$$

Uniform boundedness and approximation by range elements extend the convergence to every u. Conversely, bounded extensions restrict to (4). Strong countable additivity makes $A \mapsto P_A u$ a countably additive vector measure with bounded range for each u. The uniform boundedness principle then supplies a common bound over A and gives (10). □

This criterion states the operator property needed by the spectral projections in the proof. It is stronger than injectivity. The wrapped-Gaussian model has $C_{K,\mathcal{G}} = \infty$ for a σ-field containing the level set used after (9).

5. Full separation and the ambient boundary

Work first over real L^1. When $\mathcal{G} = \Sigma$ modulo null sets and $C = C_{K,\Sigma} < \infty$, take $A = \{g \ge 0\}$. Positivity and mass preservation give $\|K(g^+)\|_1 = \|g^+\|_1$ and the analogous equality for g^-. Hence

$$\|g\|_1 = \|P_A^0 Kg\|_1 + \|P_{A^c}^0 Kg\|_1 \le 2C\|Kg\|_1. \tag{12}$$

On the complexification, applying this argument to real and imaginary parts gives the sufficient lower bound $\|Kg\|_1 \ge (4C)^{-1}\|g\|_1$. Either bound makes $\mathcal{R}(K)$ closed. Dense range then makes K onto and K^{-1} bounded.

Assumption 4 connects full latent separation to this result. Choose a countable determining class $B_1, B_2, \ldots$ for outcome laws and define

$$q(t) = \big(P\{Y \in B_1 \mid X^* = t\}, P\{Y \in B_2 \mid X^* = t\}, \ldots\big). \tag{13}$$

On standard-Borel supports, Assumption 4 makes q a measurable injection modulo null sets, so its coordinates generate the latent σ-field. A common ambient spectral resolution compatible with all outcome-event operators therefore contains the full family KI_AK^{-1}. Proposition 1 and (12) apply.

This ambient repair conflicts with the continuous smoothing model in the article. If the conditional-density kernel satisfies $0 \le k(x,t) \le M$, then

$$\|Kg\|_\infty \le M\|g\|_1. \tag{14}$$

Surjectivity and (12) would imply $\|h\|_\infty \le 2MC\|h\|_1$ for every $h \in L^1(\nu)$. On a nonatomic support, functions concentrated on sets of shrinking measure violate this inequality. Thus a bounded scalar-type spectral repair on ambient L^1, combined with full separation and the dense-range conclusion, excludes bounded-density smoothing on a nonatomic support. Finite latent and observable supports remain within the matrix case, where a full-column-rank measurement matrix has a bounded inverse on its range.

6. Range norms and candidate comparison

For a specified measurement operator K, equip its range with the transported norm

$$\|Kg\|_K = \|g\|_1. \tag{15}$$

Then K is an isometry onto this range, $\|KM_yK^{-1}\|_K = \|m_y\|_\infty$, and $\|KI_AK^{-1}\|_K \le 1$. The latent multiplication algebra and its spectral measure are therefore available for that candidate.

Identification compares candidate factorizations. Candidates K_1 and K_2 generate different ranges and transported norms unless a common-range and equivalent-topology condition is imposed. Equality of their observable first-order operators on $\mathcal{R}(L_{x|z})$ supplies identities on a dense ambient subspace. It does not by itself make that subspace invariant under products of the outcome operators or a graph core for their closures.

The wrapped-Gaussian family gives a direct invariance failure. If H has multipliers $\ell_n = \exp(-\sigma_z^2 n^2/2)$ and the outcome multiplier has coefficients $m_n = \exp(-\sigma_y^2 n^2/2)$, the constant belongs to $\mathcal{R}(H)$. Invariance under multiplication by the outcome density would require an L^1 preimage with coefficients

$$\frac{m_n}{\ell_n} = \exp\{(\sigma_z^2 - \sigma_y^2)n^2/2\}. \tag{16}$$

For $\sigma_z > \sigma_y$, these coefficients diverge. Hence the natural observable range is not invariant. A range-based proof needs a candidate-independent graph space, a common invariant core, and a cross-candidate order isomorphism, or another device that identifies one common multiplication calculus from observables.

7. Reconstruction conditions

The next proposition gives one sufficient route. Its premises separate spectral extension from probability-kernel reconstruction.

PROPOSITION 2 (Common-resolution reconstruction). *Let the observable and latent supports be standard Borel, retain the article's conditional independence and location label, and let* $B_1, B_2, \ldots$ *determine outcome laws. Form the dense-domain event operators*

$$T_j = L_{\mathbf{1}\{Y \in B_j\};x|z} L_{x|z}^{-1}. \tag{17}$$

Suppose: (i) the T_j *extend to bounded scalar-type spectral operators on one observable Banach space* $E \hookrightarrow L^1(X)$ *and possess a common strongly countably additive joint projection measure* P*; (ii) every admissible factorization has a common invariant core on which its outcome multipliers and latent-band maps agree with this functional calculus and* P*; (iii) the joint spectral coordinate separates latent values and has multiplicity one; (iv)* $A \mapsto P(A) f_x$ *is a positive* L^1*-valued measure whose scalar mass is a probability measure and which admits nonnegative, integral-one Bochner Radon–Nikodym derivatives in the domain of the location functional; and (v) the analogous measures formed from* $P(A) f_{x|z}$ *yield absolutely continuous latent conditional laws. Then the observed law identifies the three conditional-density factors up to null sets.*

PROOF. The joint spectral coordinate recovers the conditional outcome law on the determining class. The vector measure $P(A) f_x$ and its derivative recover the measurement-density column at almost every spectral coordinate. Probability normalization sets its scale, and the location functional labels it by x^*. Applying the same projections to $f_{x|z}$ recovers the conditional law of X^* given $Z = z$. Candidate compatibility and multiplicity one make these operations common across admissible factorizations. □

The compatibility clause in Proposition 2 transfers latent bands into the observable resolution. The positivity and Radon–Nikodym clauses transfer an abstract spectral resolution back into conditional densities. These properties carry content on a nonatomic L^1 space.

A second route can be formulated only on observed positive orbits. Suppose the event operators admit a candidate-independent joint Borel calculus Φ on a common invariant domain containing f_x and every $f_{x|z}$. Let S be its joint spectrum. If $A \mapsto \Phi(\mathbf{1}_A) f_x$ is a positive countably additive L^1-valued measure with an integral-one Bochner derivative, the outcome profile maps latent values bijectively to S, the location functional labels those derivatives, and the corresponding $f_{x|z}$ measures are absolutely continuous, then the same disintegration recovers the factorization. This cyclic-orbit certificate avoids bounded projections on every signed observable direction. Its common calculus, invariant-domain, positivity, and derivative premises extend the printed assumptions.

8. Scope for identification and estimation

Under Assumptions 1–5, the wrapped-Gaussian model yields an unbounded observable similarity from a bounded latent multiplier, unbounded conjugated level-set projections, and measurement-density columns outside the inverse operator's domain. The common-resolution and cyclic-orbit propositions give sufficient routes that address these steps.

The construction supplies one probability factorization satisfying the printed assumptions. Within the translation-invariant convolution class, equality of the observable Fourier coefficients identifies that factorization up to a common shift, and the location functional fixes the shift. A theorem-level reversal requires an observationally equivalent second factorization or a separate argument producing multiple positive factorizations from the observable dense-domain family. A proof of Theorem 1 under Assumptions 1–5 requires candidate-independent uniqueness on the dense-domain graph.

The estimation supplement uses Theorem 1 to define the population target for sieve likelihood. A revised statistical theorem can add the selected reconstruction conditions to its parameter space and approximation assumptions. Finite latent-class applications can use ordinary matrix rank and simultaneous diagonalization. Continuous smoothing applications need a range, rigging, or positive-orbit argument tailored to their operators before the spectral factorization is used as an identification premise.

9. Conclusion

The observable similarity in Hu and Schennach (2008) begins on a dense range. The five printed assumptions permit smoothing models in which that similarity and its latent-band projections are unbounded on ambient L^1, and the proposed eigenfunction columns lie outside the domain. Uniform band stability exactly characterizes the bounded spectral-projection step. Combined with full outcome separation, it forces bounded invertibility and therefore excludes the continuous bounded-density smoothing case on a nonatomic support. Transported graph norms recover the algebra for one candidate, while identification calls for a common candidate-independent reconstruction. The two certificates state conditions under which spectral data return to positive conditional densities. The identification conclusion under the original five assumptions remains to be decided by a dense-domain uniqueness argument or an observational-equivalence construction.

References

Hu, Yingyao and Susanne M. Schennach. 2008. "Instrumental Variable Treatment of Nonclassical Measurement Error Models." *Econometrica* 76 (1): 195–216. `10.1111/j.0012-9682.2008.00823.x`.

Appendix A. Coordinate isometry and real operators

Define

$$(Cg)(t) = \frac{g\{\tau(t)\}}{\psi'\{\tau(t)\}}. \tag{A1}$$

Then $C : L^1(\mathbb{R}, dr) \to L^1((-\pi, \pi), dt)$ is an isometry, and the Euclidean conditional-density operators in (6) are conjugate to circular convolution. The complex exponentials in (9) can test a real operator because every bounded real operator has a bounded complexification. Equivalently, the same divergence follows from real sine and cosine pairs.

Appendix B. A conditional range-level certificate

Suppose two admissible measurement operators have the same algebraic range, their transported norms are equivalent, and $\mathcal{R}(L_{x|z})$ is a common graph core for every outcome-event similarity. Suppose the induced cross-candidate map $S = K_1^{-1}K_2$ and its inverse preserve positive cones. The observable identities extend to the common graph completion and yield $SM_{q_{2j}} = M_{q_{1j}}S$ for the determining outcome-event family. Since this family generates the latent multiplication algebra, an order isomorphism of standard L^1 spaces that conjugates it is a weighted measurable composition map. Probability normalization absorbs the weight, and the location functional fixes the remaining relabeling. These common-range, topology, core, and order conditions provide another sufficient factorization result.