Symmetry and Selection in Boundary Moment Inequalities: A Comment on Pakes, Porter, Ho, and Ishii (2015)

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Abstract

Pakes, Porter, Ho, and Ishii (2015) use symmetry of bank-specific cost shocks to construct boundary moment inequalities. The construction evaluates a sign-reversing order-statistic map at the realized number of positive choices. That count can depend on the same shocks, leaving the selected statistic's expectation unrestricted under marginal iid symmetry. A three-agent ordered-choice model satisfies the maintained choice, counterfactual, information, and symmetry conditions and yields a negative population moment at the true parameter. Joint reflection of shocks and the selected count restores finite-sample validity; a convergent selected share yields a separate large-market result. In the ATM application, the lower estimate remains 24,452 dollars; the upper estimates of 26,444 and 26,644 dollars require a repaired condition. The correction applies to the boundary sufficient-condition argument and specifications using it. The general moment inequality continues to hold when Assumptions 1–3 are imposed directly.

JEL Codes: C12, C14, C57, L13

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1. Introduction

Pakes et al. (2015) show how optimizing behavior can generate moment inequalities when profits are measured with error and agents possess information unavailable to the econometrician. Their ordered-choice application allows the decision d_i to equal zero. At this boundary, the left comparison $d_i - 1$ is infeasible, and the paper uses an iid symmetry condition on the structural cost shocks to replace the missing comparisons.

The replacement is formed market by market. Let n_L be the number of agents with $d_i > 0$. The proof orders the cost shocks and obtains the statistic

$$G_{n_L}(\eta) = \sum_{i=1}^{n_L} \eta_{(i)} - \sum_{i=1}^{n-n_L} \eta_{(i)}. \tag{1}$$

For every fixed integer k, reflection of the shock vector changes the sign of G_k. In the published construction, n_L is a realized choice count. Decisions may depend on the structural cost shocks and the revenue schedules, so symmetry of the marginal shock distribution does not determine the joint distribution of (η, n_L).

The argument begins with the boundary step in the version of record and then considers a three-agent ordered-choice model in which the proposed observable moment has negative expectation at the true parameter. The construction retains a latent cost component after conditioning on the observed revenue schedules. This leads to two possible repairs: a finite-market condition that validates the reflection argument and a large-market approximation based on a convergent selected share. The final sections apply the distinction to the reported ATM estimates and trace its use in later work.

The result concerns the sufficient-condition argument used to verify Assumption 3(a) for the boundary weights on pages 325–326 of Pakes et al. (2015). If Assumptions 1–3 are imposed directly, the implication leading to the paper's equation (5) remains in force. Table I already identifies which ATM endpoints use the boundary equality. Re-estimation under an alternative boundary construction requires the bank-level revenue comparisons and the program that forms and resamples the market moments.

2. The published boundary construction

In the ordered-choice model of Pakes et al. (2015), the unit cost is $\theta_0 + \eta_i$, where the bank-specific component η_i is known to the bank when it chooses d_i. The structural disturbance in a comparison between d and d' is

$$\nu_{2,i,d,d'} = (d' - d)\eta_i. \tag{2}$$

The boundary section assumes that the η_i are iid and symmetrically distributed about zero.

Within a market of n agents, define

$$L = \{i : d_i > 0\}, \qquad n_L = |L|. \tag{3}$$

The proof introduces the latent sets containing the lowest n_L shocks and the lowest $n - n_L$ shocks,

$$L_\eta = \{i : \eta_i \le \eta_{(n_L)}\}, \qquad U_\eta = \{i : \eta_i \le \eta_{(n-n_L)}\}. \tag{4}$$

An observable rank set formed from the right revenue comparisons contains a bound based on U_η. After the best-response and counterfactual inequalities are applied, the structural component of the selected moment is bounded below by $n^{-1}G_{n_L}(\eta)$, where

$$G_k(e) = \sum_{i=1}^{k} e_{(i)} - \sum_{i=1}^{n-k} e_{(i)}. \tag{5}$$

Page 325 of the 2015 Econometrica version of record assigns this term mean zero from iid symmetry. The December 2011 longer version and the July 2014 author manuscript define the second latent set with an upper-tail inequality. Every statement below concerns the lower-tail definition printed in the version of record.

LEMMA 1 (Fixed-count reflection). *For every vector $e \in \mathbb{R}^n$ and fixed $k \in \{0, \dots, n\}$,*

$$G_k(-e) = -G_k(e). \tag{6}$$

PROOF. The order statistics of the reflected vector satisfy $(-e)_{(i)} = -e_{(n+1-i)}$. Substitution in (5) gives

$$G_k(-e) = -\sum_{i=n-k+1}^{n} e_{(i)} + \sum_{i=k+1}^{n} e_{(i)} = -G_k(e). \tag{7}$$

The last equality follows by cancelling the overlapping ranges, with the same calculation applying on either side of $k = n/2$. □

If k is fixed and the distribution of E is centrally symmetric, Lemma 1 gives $\mathbb{E}G_k(E) = 0$. With a selected count K, reflection gives $G_K(-E) = -G_K(E)$ while the reflected economy can have a different count. The expectation used in the boundary proof therefore depends on the coupling between E and K.

3. A three-agent ordered-choice model

The next proposition embeds the selected-count calculation in a choice model with the information and counterfactual structure used by Pakes et al. (2015).

PROPOSITION 1 (Boundary moment at the true parameter). *There is a three-agent ordered-choice model with iid cost shocks symmetric about zero, unique optimal decisions, fixed revenue schedules under unilateral deviations, and symmetric information for which the boundary moment proposed by Pakes et al. (2015) has strictly negative expectation at θ_0.*

PROOF. Let η_1, η_2, η_3 be iid uniform random variables on $[-1, 1]$. Let K equal the number of negative shocks, clipped to the two interior values:

$$K = \min\{\max\{\#\{i : \eta_i < 0\}, 1\}, 2\}. \tag{8}$$

Assign $s_i = 1$ to the K agents with the lowest shocks and $s_i = 0$ to the remaining agents. Then $K = 1$ when $\eta_{(2)} \geq 0$ and $K = 2$ when $\eta_{(2)} < 0$, so

$$G_K(\eta) = -|\eta_{(2)}|. \tag{9}$$

Set $\epsilon = 1/16$ and first consider the revenue schedule

$$R_i(d) = d\eta_i - \epsilon(d - s_i)^2 + \epsilon s_i, \qquad d \in \mathbb{Z}_+. \tag{10}$$

At $\theta_0 = 0$, profit equals $R_i(d) - d\eta_i = -\epsilon(d - s_i)^2 + \epsilon s_i$, and $d_i = s_i$ is the unique optimum. For a selected agent, the left revenue comparison is $\eta_i + \epsilon$. For every agent, the right comparison at the observed choice is $-\eta_i + \epsilon$. Ranking the right comparisons therefore selects the $3 - K$ lowest cost shocks. The observable boundary moment is

$$\frac{1}{3}G_K(\eta) + \epsilon = -\frac{|\eta_{(2)}|}{3} + \epsilon. \tag{11}$$

The median of three iid uniform variables on $[-1, 1]$ has density $3(1-x^2)/4$. Hence $\mathbb{E}|\eta_{(2)}| = 3/8$, and the population moment in (11) equals $-1/16$.

Continuous perturbations retain the strict inequality and leave the structural cost outside the observed vector. Draw independent $u_i, v_i \sim U[0, \delta]$ and define the revenue increments by

$$R_i(0) = 0, \tag{12}$$

$$R_i(1) - R_i(0) = \eta_i + (2s_i - 1)(\epsilon + u_i), \tag{13}$$

$$R_i(d+1) - R_i(d) = \eta_i - \epsilon - v_i - (d - 1), \qquad d \geq 1. \tag{14}$$

For $s_i = 0$, every profit increment is negative. For $s_i = 1$, the first profit increment is positive and each later increment is negative. Thus $d_i = s_i$ remains the unique global optimum. The perturbations can raise the market moment by at most δ. Taking $\delta = 1/128$ gives

$$\mathbb{E}m(W, \theta_0) \leq -\frac{1}{16} + \frac{1}{128} = -\frac{7}{128} < 0. \tag{15}$$

Let each agent's information set contain the state and all cost shocks, and let the revenue schedule remain fixed when that agent contemplates a unilateral change in d_i. Set the expectational and measurement disturbance ν_1 to zero and use the constant instrument. These assignments give the best-response, counterfactual, and symmetric-information conditions used in the boundary derivation. To verify that η_i remains latent, write $A_i = R_i(1) - R_i(0)$ and $B_i = R_i(2) - R_i(1)$. Conditional on the observed schedule and decision, the compatible values of η_i contain an interval of positive length: for $s_i = 1$ they lie in

$$[A_i - \epsilon - \delta, A_i - \epsilon] \cap [B_i + \epsilon, B_i + \epsilon + \delta], \tag{16}$$

and for $s_i = 0$ they lie in

$$[A_i + \epsilon, A_i + \epsilon + \delta] \cap [B_i + \epsilon, B_i + \epsilon + \delta]. \tag{17}$$

The realized value lies in the interior almost surely. Later increments are determined by B_i and add no equation for the shock. Small movements in these intervals preserve the strict shock order, signs, count, and observed decisions almost surely. The boundary assumptions impose iid symmetry on the marginal cost shocks and permit their dependence with the revenue schedules that agents observe. The constructed joint law therefore lies in the stated boundary model and yields (15). □

TABLE 1. Maintained conditions in the three-agent model

Condition used in the boundary derivation	Realization in the three-agent model
Ordered nonnegative choice	$d_i = s_i \in \{0, 1\}$ is the unique global maximizer of profit under the perturbed revenue increments.
Unilateral counterfactual	The realized revenue schedule is held fixed while agent i compares d_i with $d_i - 1$ and $d_i + 1$.
Agent information	Each agent observes the state and all cost shocks; the expectational and measurement disturbance is set to zero.
Marginal shock restriction	The η_i are iid uniform on $[-1, 1]$ and hence symmetric about zero.
Latent cost after observables	Conditional on the observed decision and revenue comparisons, equations (16)–(17) leave a positive-length interval of compatible η_i values.
Boundary rank operation	The baseline right comparisons select the complementary lower order statistics and yield $G_K(\eta)/3 + \epsilon$; the continuous perturbation changes the population moment by at most δ.

The table matches the objects used in the boundary sufficient-condition argument. The general implication from Assumptions 1–3 is outside the proposition's claim.

The counterexample uses interior counts in every state. Empty-set conventions at $K = 0$ or $K = n$ play no role. Its negative expectation arises before sampling variation or an estimated parameter enters the calculation.

4. Finite-market validity

The fixed-count identity extends to a random count under a condition on the joint law.

PROPOSITION 2 (Joint reflection). *Let $E \in \mathbb{R}^n$ and $K \in \{0, \ldots, n\}$. If $\mathbb{E}|G_K(E)| < \infty$ and*

$$(E, K) \overset{d}{=} (-E, K), \tag{18}$$

then $\mathbb{E}G_K(E) = 0$.

PROOF. Joint reflection and Lemma 1 give

$$\mathbb{E}G_K(E) = \mathbb{E}G_K(-E) = -\mathbb{E}G_K(E). \tag{19}$$

Integrability yields the result. □

Condition (18) holds when the conditional distribution of E given $K = k$ is centrally symmetric for every count with positive probability. Independence of K and a centrally symmetric E is one primitive sufficient condition. A count determined by reflection-invariant features, including absolute order statistics, also preserves the required coupling. These conditions identify what the fixed-count calculation needs once the count is chosen within the model.

The selected-count effect can have a fixed sign. If $K(E) = \#\{i : E_i < 0\}$ and ties occur with probability zero, then

$$G_{K(E)}(E) \le 0 \tag{20}$$

for every realization. When $K(E) \le n - K(E)$, the statistic equals the negative sum of the nonnegative central order statistics. When $K(E) > n - K(E)$, it equals the sum of the negative central order statistics. The three-agent identity (9) is the smallest interior-count instance.

5. A large-market result

A deterministic limit for the selected share can restore the cancellation asymptotically even when finite-market joint reflection is unavailable.

PROPOSITION 3 (Convergent selected share). *Let $E_1, E_2, \dots$ be iid with continuous distribution function F, finite first moment, and symmetry about zero. Let K_n satisfy*

$$\frac{K_n}{n} \xrightarrow{p} q \in [0,1]. \tag{21}$$

If the empirical quantile integrals converge at q and $1-q$, then

$$\frac{1}{n} G_{K_n}(E_1, \dots, E_n) \xrightarrow{p} 0. \tag{22}$$

PROOF. The two order-statistic sums converge to

$$\frac{1}{n}\sum_{i=1}^{K_n} E_{(i)} \xrightarrow{p} \int_0^q F^{-1}(u)\,du, \qquad \frac{1}{n}\sum_{i=1}^{n-K_n} E_{(i)} \xrightarrow{p} \int_0^{1-q} F^{-1}(u)\,du. \tag{23}$$

Symmetry gives $F^{-1}(1-u) = -F^{-1}(u)$ at continuity points. Since the mean is zero,

$$\int_0^{1-q} F^{-1}(u)\,du = -\int_{1-q}^{1} F^{-1}(u)\,du = \int_0^{q} F^{-1}(u)\,du. \tag{24}$$

Their difference converges to zero. □

Propositions 2 and 3 apply to different sampling sequences. Joint reflection supports repeated markets with a fixed number of agents. The selected-share result concerns growth in the number of agents within a market. Adding fixed-size markets averages the finite-market expectation and preserves any selection component in that expectation.

6. The ATM estimates

The ATM application contains 291 banks in ten local markets. Table I of Pakes et al. (2015) reports six specifications. The odd-numbered rows form the upper moment from banks with positive ATM choices. The even-numbered rows include the zero-choice banks through the order-statistic construction in Section 2. Table 2 shows how the correction changes the interpretation of the values printed in the article.

TABLE 2. ATM estimates reported in Table I and their interpretation

Row	Specification	Published output	Implication of the correction
1	Constant; positive choices	[24,452, 25,283]	Upper endpoint conditions on $d_i > 0$ and retains the selection problem described in the article.
2	Constant; boundary correction	[24,452, 26,444]	Lower endpoint remains 24,452; the finite upper endpoint requires a condition such as (18).
3	Constant and three instruments; positive choices	Criterion minimizer 19,264	Sample moments have an empty intersection; the upper moments retain the positive-choice selection.
4	Constant and three instruments; boundary correction	Criterion minimizer 20,273	Boundary moments use the selected-count equality; the article's specification test rejects this specification.
5	One- and two-ATM deviations; positive choices	[24,452, 25,283]	Upper endpoint conditions on $d_i > 0$ and retains the selection problem described in the article.
6	One- and two-ATM deviations; boundary correction	[24,452, 26,644]	Lower endpoint remains 24,452; the finite upper endpoint requires a condition such as (18).

The lower moment compares each observed choice with an additional ATM. It is available for every bank, and the latent contribution to the observable lower moment is $-\eta_i$, whose expectation is zero under the marginal symmetry assumption. The constant-instrument lower estimate of 24,452 dollars in rows 1, 2, 5, and 6 therefore continues to apply. The upper moment compares an observed choice with one fewer ATM. Rows 2 and 6 supply the missing comparisons at $d_i = 0$ through $G_{n_L}(\eta)$, so their finite upper estimates use the expectation addressed by Proposition 1.

Let $\overline{\Theta}$ denote the upper endpoint of the parameter space used for the cost parameter.

Under the published boundary assumptions, the constant-instrument identified interval supported by these moments is therefore

$$[24{,}452, \overline{\Theta}]. \tag{25}$$

The numerical value of $\overline{\Theta}$ depends on the parameter space specified for estimation. The value 25,283 from rows 1 and 5 uses positive-choice banks for its left comparison; the article introduces the boundary correction because this selection can exclude banks with high ATM costs. Its reported value diagnoses the positive-choice subsample. A population upper endpoint under the published model requires a condition that signs the boundary moment.

If the within-market law satisfies joint reflection (18), the boundary equality holds and the published intervals $[24{,}452, 26{,}444]$ and $[24{,}452, 26{,}644]$ retain their stated interpretation. This condition concerns the joint law of latent costs and the realized count, so the ATM observables alone do not select between that repaired model and the model in Proposition 1. The published confidence interval $[20{,}472, 30{,}402]$ for row 2 likewise pertains to the repaired model. A confidence interval under another boundary construction requires recomputing the sample moments and their resampling distribution from the bank-level revenue differences.

7. Scope and use

Rows 2, 4, and 6 of Table I include banks at the zero-ATM boundary and use the symmetry correction. The paper's discussion identifies row 2 as the correction of truncation bias from banks with $d_i = 0$. Proposition 1 shows that the stated marginal symmetry condition leaves the population moment underlying those rows without a sign. Section 6 gives the resulting interval from the reported constant-instrument moments. Alternative finite upper estimates require the bank-level inputs and the program that forms the market moments.

The boundary construction has also been carried into later work. Kline, Pakes, and Tamer (2021) describe the ATM boundary problem, invoke iid symmetry, and direct readers to Pakes et al. (2015) for the construction. Kireyev (2020) use the order-statistic correction for nonentrants in a model of ideation contests, and Starc and Wollmann (2025) use an analogous selected-count correction for market entry in generic drugs. Their empirical conclusions depend on their own payoff structures, instruments, and sampling sequences. The relevant question in each application is whether the realized count satisfies joint reflection, a large-within-market approximation, or another condition that signs the selected boundary term.

Some applications restrict positive choices to outnumber boundary observations. This count relation leaves the coupling issue open. Appendix A gives a five-agent selected-count construction with $K > 5 - K$ in every state and a negative population boundary moment.

The correction leaves the general logic of moment inequalities intact. When the disturbance conditions in Assumption 3 of Pakes et al. (2015) hold, the observable moment follows. At a one-sided boundary, marginal symmetry of structural costs supplies those conditions only through an additional joint or asymptotic restriction of the type stated above.

8. Conclusion

The boundary statistic in Pakes et al. (2015) is odd under shock reflection at each fixed choice count. The realized count belongs to the economic model and can move under the same reflection. A condition on the joint law of shocks and counts is therefore part of the finite-market argument. The three-agent model shows that the published marginal symmetry condition permits a negative observable moment at the true parameter. Joint reflection repairs the finite-market result, while convergence of the selected share yields a separate large-market route. In the ATM application, the all-bank right comparison retains the lower estimate of 24,452 dollars, and the selected-count correction supplies no finite upper endpoint under the published assumptions. The joint-law conditions determine which empirical specifications can use order statistics to replace infeasible comparisons at a choice boundary.

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Appendix A. A count-dominance construction

Let $E_1, \ldots, E_5$ be iid uniform random variables on $[-1, 1]$. At counts three and four, equation (5) becomes

$$G_3 = E_{(3)}, \qquad G_4 = E_{(2)} + E_{(3)} + E_{(4)}. \tag{A1}$$

Choose $K = 3$ when $G_3 \leq G_4$ and $K = 4$ otherwise. Then $K > 5 - K$ in every state and $G_K = \min\{G_3, G_4\}$. Each fixed-count statistic has expectation zero by Lemma 1. Since $G_3 - G_4 = -(E_{(2)} + E_{(4)})$ and direct integration gives $\mathbb{E}|E_{(2)} + E_{(4)}| = 1/2$,

$$\mathbb{E}G_K = \frac{\mathbb{E}G_3 + \mathbb{E}G_4 - \mathbb{E}|G_3 - G_4|}{2} = -\frac{1}{4}. \tag{A2}$$

Assign the lowest K agents to the positive-choice group and use the revenue construction in Proposition 1 with $\epsilon = 1/32$. The population boundary moment is

$$\frac{1}{5}\mathbb{E}G_K + \epsilon = -\frac{3}{160}. \tag{A3}$$

Perturbations bounded by $1/256$ retain a negative upper bound of $-19/1280$ and the interval argument for the latent costs. Thus the count inequality used in the downstream contest application does not supply the reflection condition in Proposition 2.