THE COUNTERFACTUAL RECONSTRUCTION THEOREM

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WHAT IS THE ECONOMIC QUESTION?

Do locally valid counterfactual states arise from one economic history?

- One experiment identifies a minimal state on its own policy domain.
- Overlap records make neighboring states agree.
- A common history must generate every local state at once.

WHAT CAN LOCAL AGREEMENT MISS?

Context 1: (x, y) Context 2: (y, z) Context 3: (z, x)

Pairwise agreement leaves every $(x, y, z) \in \mathbb{R}^3$ feasible.

Theorem (One common history)
The globally generated states satisfy

$$\boxed{z = x + y}.$$

The remaining scalar $z - x - y$ records the missing global restriction.

WHICH ORDER OF OPERATIONS DEFINES THE PROBLEM?

$$\underbrace{\Gamma(\operatorname{im} T)}_{\mathcal{L}\text{ glue local minima}} \supseteq \underbrace{\operatorname{im}\Gamma(T)}_{\mathcal{M}\text{ minimize after gluing histories}}.$$

- T maps histories into the declared future responses.
- $\mathcal{L}$ contains compatible local response states.
- $\mathcal{M}$ contains responses generated by one common history.

WHERE DOES THE MISSING STATE LIVE?

$$\boxed{\mathfrak{O} = \mathcal{L}/\mathcal{M}}$$

$\mathcal{L}$	every locally minimal response state that agrees on observed overlaps
$\mathcal{M}$	the response states produced by one global history
$\mathfrak{O}$	the directions left after common-history responses are removed

The future-response family enters before these spaces are formed.

WHAT DOES THE RECONSTRUCTION THEOREM IDENTIFY?

Theorem (Counterfactual reconstruction)

The short exact sequence

$$0 \to \mathcal{K} \to \mathcal{H} \to \mathcal{Z} \to 0$$

induces

$$\Gamma(\mathcal{H}) \to \mathcal{L} \xrightarrow{\partial} H^1(\mathcal{K}) \to H^1(\mathcal{H}),$$

with

$$\ker \partial = \mathcal{M}, \qquad \mathfrak{O} \cong \operatorname{im} \partial.$$

A local atlas has a common-history lift precisely when its connecting class is zero.

WHY IS ONE DIAGNOSTIC MAP COMPLETE?

Let $A : \mathcal{L} \to \mathbb{R}^m$ vanish on every common-history response.

$$\mathcal{M} \subseteq \ker A \quad \Longrightarrow \quad A = \bar{A}\,\pi$$

for a unique map $\bar{A} : \mathfrak{O} \to \mathbb{R}^m$ and the quotient $\pi : \mathcal{L} \to \mathfrak{O}$. Since $\mathfrak{O} \cong \operatorname{im} \partial$, every valid linear specification restriction factors through the connecting class.

A complete diagnostic needs exactly $\dim \mathfrak{O}$ coordinates.

WHAT DOES THE BASELINE ARCHIVE LEAVE UNRESOLVED?

Let $P : \mathfrak{O} \to D$ collect obstruction information already stored in the archive.

$$\boxed{N = \ker P}$$

N contains the common-history directions still hidden after the baseline records are read. An additional record is useful only through its restriction to N.

WHAT DOES A DOWNSTREAM TARGET SELECT?

For a declared counterfactual or welfare target $G : \mathfrak{O} \to W$, define

$$\boxed{\mathfrak{Q}_{G,P} = N/(N \cap \ker G)}.$$

- States in $N \cap \ker G$ change neither the archive nor the target.
- Every identifying augmentation maps onto $\mathfrak{Q}_{G,P}$.
- The map $[n] \mapsto Gn$ embeds this quotient in the target space.

This is the target bridge state carried by any sufficient archive augmentation.

WHEN DO SELECTED RECORDS IDENTIFY THE TARGET?

Let A_S stack the chosen bridge records and $R_S = (P, A_S)$.

Theorem (Archive-relative identification)

The following statements are equivalent:

$$R_S u = R_S u' \implies Gu = Gu',$$

$$\ker(A_S|_N) \subseteq \ker(G|_N),$$

$$\operatorname{im}((G|_N)^*) \subseteq \operatorname{span}\{a_j|_N : j \in S\}.$$

HOW MANY SCALAR RECORDS ARE ENOUGH?

$$\boxed{r_G(P) = \operatorname{rank}(G|_{\ker P}) = \dim \mathfrak{Q}_{G,P}}$$

Every identifying linear record system has at least $r_G(P)$ coordinates. A basis of the missing target row space attains the bound.

Full reconstruction is the case $G = I_{\mathfrak{H}}$.

WHY DOES TARGET IDENTIFICATION MATTER FOR DECISIONS?

For every finitely supported law on the obstruction, every finite action set, and every bounded loss depending on Gu,

$$\mathrm{Risk}(R_S) = \mathrm{Risk}(G)$$

holds precisely when the selected archive identifies G. If identification fails, two states share one archive record and imply distinct targets. A two-action problem then gives

$$\mathrm{Risk}(G) = 0, \qquad \mathrm{Risk}(R_S) = \frac{1}{2}.$$

The bridge quotient preserves the value of target observation across finite decision problems.

HOW DO SEVERAL TARGETS SHARE BRIDGE RECORDS?

For target family $\{G_i : i \in I\}$, stack the targets in $S \subseteq I$ and define

$$\rho_P(S) = \operatorname{rank}(G_S|_N).$$

Theorem (Target portfolio)

ρ_P is normalized, increasing, and submodular:

$$\rho_P(S) + \rho_P(T) \geq \rho_P(S \cup T) + \rho_P(S \cap T).$$

The overlap of missing target row spaces is the exact dimension of reusable bridge information.

WHAT IS THE PRICE OF RECORD ERROR?

Suppose an identifying archive record is observed with $\|e\| \leq \varepsilon$.

$$\boxed{\inf_{\delta} \sup_{u,e} \|\delta(R_S u + e) - Gu\| = \varepsilon \|G R_S^{\dagger}\|_{\mathrm{op}}}$$

The pseudoinverse decoder attains the bound. The operator norm

measures how record error is amplified into target error.

WHAT REMAINS UNDER A SMALLER RECORD BUDGET?

Let $\lambda_1(G,P) \geq \lambda_2(G,P) \geq \cdots$ be the eigenvalues of residual target covariance after linear projection on the baseline archive.

$$\boxed{\mathcal{R}^\star_m(G,P) = \sum_{j>m} \lambda_j(G,P)}$$

The first m principal coordinates of the residual target attain the frontier.

exact recovery	the tail reaches zero
record value	record m is worth $\lambda_m(G,P)$
effective dimension	small trailing eigenvalues measure the cost of compression

HOW DO NOISE AND PRECISION CHANGE THE DESIGN?

Under unit residual-signal normalization and noise variance σ^2,

$$\mathcal{R}^{\star}_{m,\sigma^2} = \sum_j \lambda_j - \frac{1}{1+\sigma^2}\sum_{j=1}^{m} \lambda_j.$$

With precision budget $\sum_j \tau_j \le b$,

$$\tau_j^{\star}(b) = \left(\sqrt{\frac{\lambda_j}{\nu(b)}} - 1\right)_+ .$$

The record directions remain the residual target eigen-directions. Precision

follows their target value.

HOW CAN THE COMMON-HISTORY RESTRICTION BE TESTED?

Choose coordinates $A^\star : \mathcal{L} \to \mathbb{R}^o$ for a complete diagnostic, so $\mathcal{M} = \ker A^\star$.
If $\sqrt{n}(\widehat{z} - z_0) \Rightarrow N(0, \Sigma)$ and $z_0 \in \mathcal{M}$, then

$$n(A^\star \widehat{z})^\top \widehat{\Omega}^{-1}(A^\star \widehat{z}) \Rightarrow \chi^2_o, \qquad \Omega = A^\star \Sigma A^{\star\top}.$$

The test degrees of freedom equal the dimension of a complete

reconstruction diagnostic.

WHY IS EXACT RANK UNSTABLE NEAR A BOUNDARY?

In the Gaussian matrix experiment, compare a rank-r matrix A_0 with

$$A_{n,h} = A_0 + \frac{h}{\sqrt{n}} uv^*.$$

Every rank estimator satisfies

$$\Pr_{A_0}(\widehat{r}_n \neq r) + \Pr_{A_{n,h}}(\widehat{r}_n \neq r+1) \geq 2\{1 - \Phi(h/2)\}.$$

A root-n singular direction changes the exact record count while remaining

statistically contiguous to the boundary.

WHAT CAN BE REPORTED AT A RANK BOUNDARY?

An operator-norm confidence radius c_n yields

$$\#\{j : \sigma_j(\widehat{A}) > \tau + c_n\} \le r_\tau(A) \le \#\{j : \sigma_j(\widehat{A}) \ge (\tau - c_n)_+\}$$

simultaneously over every resolution τ. A primitive confidence region C_n

yields the projected target set

$$\mathcal{C}_{G,n} = \overline{\bigcup_{\vartheta \in C_n} \mathcal{I}_G(\vartheta)}.$$

The procedure carries all ranks admitted by the primitive data into the counterfactual target.

HOW ARE THE BOUNDARY CRITICAL VALUES COMPUTED?

For $\widehat{A} = A + \sigma n^{-1/2} Z$ with Gaussian Z,

$$c_{n,\alpha} = \frac{\sigma q_{p,q}(1-\alpha)}{\sqrt{n}}, \qquad q_{p,q}(1-\alpha) = Q_{1-\alpha}(\|Z\|_{\mathrm{op}}).$$

For $\widehat{\theta} \sim N_d(\theta, V/n)$,

$$C_{n,\alpha} = \{\vartheta : n(\widehat{\theta} - \vartheta)^\top V^{-1}(\widehat{\theta} - \vartheta) \leq \chi^2_{d,1-\alpha}\}.$$

Both confidence events have finite-sample coverage $1-\alpha$ under the stated Gaussian experiments.

WHAT DOES THE OREGON APPLICATION RECONSTRUCT?

For reception group g, let

$$a_g = \Pr(\text{application} \mid \text{invitation}, g), \quad b_g = \Pr(\text{approval} \mid \text{application}, g),$$

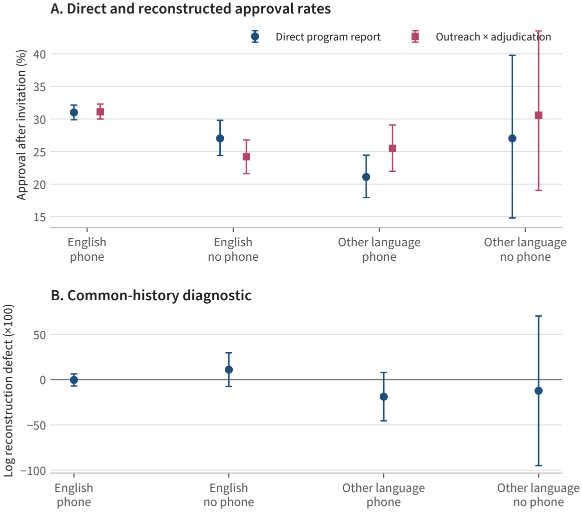
$$d_g = \Pr(\text{approval} \mid \text{invitation}, g).$$

A common administrative history implies

$$\delta_g = \log d_g - \log a_g - \log b_g = 0.$$

Three disjoint household extracts estimate the three local reports within eight lottery draws.

DO THE THREE REPORTS FIT ONE COMMON HISTORY?



Household bootstrap within lottery draw and reporting extract. Panel B uses one simultaneous max-t critical value.

HOW DOES THE TARGET CHANGE THE ARCHIVE REQUIREMENT?

Declared target	Estimate	Records
Lottery effect on enrollment	25.6 [24.9, 26.3] pp	0
Approval after invitation, population	29.2 [28.6, 29.8]%	1
Approval after invitation, four groups	23.3–30.4%	4; set $\{4\}$

The evaluated archive retains the lottery–enrollment experiment and conditional

approval reports. The target selects which upstream application-score directions must cross the reporting handoff.

WHERE DOES THE CONTRIBUTION SIT?

Imported result	Role in the argument
sheaf cohomology	connecting map and global lift obstruction
minimal realization	image as the smallest response state
functional observability	target recovery from selected records
Blackwell comparison	value across finite decision problems
spectral design	residual risk and precision allocation
rank inference	boundary-aware confidence procedures

The economic response morphism derives the obstruction. The baseline archive

and downstream target then determine the bridge quotient on which these tools operate.

WHAT SHOULD THE READER CARRY FORWARD?

1. Local response states form one economy when their connecting class vanishes.
2. A baseline archive and downstream target select a canonical bridge quotient.
3. Its dimension is the attainable record count; its spectrum prices partial recovery.
4. Boundary inference carries rank uncertainty into the final target.

TECHNICAL APPENDIX

WHICH OBJECTS DEFINE THE THEOREM?

$\mathcal{H}$	sheaf of locally available histories
$\mathcal{Y}$	sheaf of declared future response profiles
T	response morphism from histories to future responses
$\mathcal{K} = \ker T$	locally silent history directions
$\mathcal{Z} = \operatorname{im} T$	locally minimal response state
$\mathcal{L} = \Gamma(\mathcal{Z})$	compatible local atlas
$\mathcal{M} = \operatorname{im} \Gamma(T)$	responses generated by a common history
$\mathfrak{O} = \mathcal{L}/\mathcal{M}$	reconstruction obstruction

WHICH STATEMENTS DOES THE MAIN THEOREM JOIN?

- $\mathfrak{O} \cong \operatorname{im} \partial = \ker[H^1(\mathcal{K}) \to H^1(\mathcal{H})]$.
- $\mathcal{M}$ is the universal minimal global response state.
- Every valid diagnostic factors uniquely through $\mathfrak{O}$.
- A target is identified exactly when selected records span its missing row space.
- $\mathfrak{Q}_{G,P}$ is the universal target bridge state and $\dim \mathfrak{Q}_{G,P} = r_G(P)$.
- Bounded error, record budgets, noise, and precision act on the same target quotient.

HOW DOES THE PROOF REDUCE TO TWO QUOTIENT ARGUMENTS?

Global lift

$$\mathcal{L}/\mathcal{M} \cong \operatorname{im} \partial$$

Archive-relative target

$$\ker P/(\ker P \cap \ker G) \cong \operatorname{im}(G|_{\ker P})$$

The first quotient classifies common-history failure. The second retains the part of that failure that reaches the declared target after the archive is read.

HOW DOES THE THREE-CONTEXT MATRIX EXAMPLE WORK?

$$Q(u, v) = (u, v, v, u + v, u + v, u)^{\top},$$

$$\partial(x, y, z) = z - x - y.$$

The overlap matrix has rank three, so $\dim \mathcal{L} = 3$. The global response map has rank two, so $\dim \mathcal{M} = 2$.

$$\dim \mathfrak{O} = \dim \mathcal{L} - \dim \mathcal{M} = 1.$$

Each local two-coordinate response remains free even though the common global history imposes one cross-context restriction.

WHAT DOES THE RANK-BOUNDARY SIMULATION SHOW?

matrix dimension	3×3
sample sizes	100, 400, 1600
local singular shifts	0, 0.5, 1, 2, 4 in root-n units
economic resolutions	0, 0.05, 0.10
draws	6,000 per configuration; 12,000 independent calibration draws
minimum bracket coverage	0.9923 across 45 configurations
Gaussian primitive ellipsoid	0.9476 over 20,000 draws at nominal 0.95

HOW IS THE OREGON EVIDENCE REPRODUCED?

source	NBER Oregon Health Insurance Experiment public-use files
source DOI	10.60592/rx0z-sp89
common-history sample	29,799 people in 24,901 households; eight lottery draws
report split	stable household hash within draw; three disjoint extracts
inference	4,000 household bootstrap draws; simultaneous max-t across four groups
archive experiment	full linked access statistics plus lottery-enrollment sufficient statistics

The replication script reads the processed data after matching both recorded

hashes.

WHERE DO THE REGULAR EXTENSIONS STOP?

- At a regular history, constant rank gives local state dimension rank DQ_h and tangent obstruction dimension $\dim T_z\mathcal{L} - \operatorname{rank} DQ_h$.
- A global smooth state exists under the closed embedded submersive equivalence-relation criterion.
- Rank-changing images and non-Hausdorff quotients require separate singular theories.

The finite linear theorem remains the source of the paper's exact identification and acquisition results.

WHICH REFERENCES ANCHOR THE CONSTRUCTION?

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- Abramsky and Brandenburger (2011), *New Journal of Physics*; Wu, Xie, and Li (2026), arXiv:2603.17384.
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