ON COFACTUALS

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WHICH POLICY OBJECT REQUIRES A CROSS-WORLD PAIRING?

A policy gain compares outcomes for the same unit across two regimes.

$$G_{S,T} = Y_S - Y_T$$

- Separate samples can reveal the marginal laws of Y_S and Y_T.
- The law of $G_{S,T}$ also depends on how those outcomes move together.
- The paper studies a shared-component pairing whose loadings come from the policy labels.

HOW CAN THE SAME MARGINALS PRODUCE DIFFERENT GAINS?

Suppose Y_S, Y_T, and $Y_{S\cap T}$ all have the standard normal law.

Pairing	Construction	Gain variance
Shared source	$Y_S = U, Y_T = U$	0
Independent pairing	$Y_S = U_S, Y_T = U_T$	2

The marginal evidence agrees. The coupling selects the distribution of the policy gain.

WHICH COORDINATES ENCODE THE POLICY SYSTEM?

Worlds are indexed by policy sets $S \subseteq \mathcal{P}$. Möbius inversion defines

$$C_A = \sum_{B \subseteq A} (-1)^{|A|-|B|} Y_B, \qquad Y_S = \sum_{A \subseteq S} C_A.$$

- $C_\varnothing$ records the baseline component.
- $C_{\{j\}}$ records the increment attached to policy label j.
- C_A for $|A| \geq 2$ records the interaction that enters worlds containing every label in A.

WHEN DO THE MARGINALS ADMIT A FACTORIZED COUPLING?

Theorem (Marginal-to-joint characterization)
Under local exponential integrability, a factorized coupling exists if and only if every Möbius-transformed cumulant generating function is an admissible CGF. Within the class, the component laws and the joint law are unique.

The policy labels fix the loading matrix. Admissibility supplies the probability laws carried by its columns.

HOW DOES THE THEOREM CONSTRUCT THE JOINT LAW?

For each component label A, form

$$H_A(t) = \sum_{B \subseteq A} (-1)^{|A|-|B|} K_B(t).$$

1. Interpret each admissible H_A as the CGF of C_A.
2. Join the component laws independently.
3. Assemble every world through $Y_S = \sum_{A \subseteq S} C_A$.

The resulting law matches the supplied world marginals and fixes their joint law within the class.

HOW CAN TRANSFORM ADMISSIBILITY BE CHECKED?

Write $\Lambda_A(t) = \exp\{H_A(t)\}$. A candidate component law exists exactly when

$$\left[\Lambda_A(t_i + t_j)\right]_{i,j=1}^{m} \succeq 0$$

for every finite grid in the common transform interval, together with continuity and $\Lambda_A(0) = 1$.

- The condition is a hierarchy of positive-semidefinite restrictions.
- A finite negative eigenvalue supplies a finite incompatibility witness.
- Expanding grids turn the population characterization into a consistent sample check.

WHAT FINITE-SAMPLE GUARANTEE ACCOMPANIES THE PSD CHECK?

For an empirical grid matrix $\widehat{\Gamma}_{A,N}$, reject when

$$\min_A \lambda_{\min}(\widehat{\Gamma}_{A,N}) < -m_N \Delta_{N,\alpha}.$$

- Known bounded support gives $\Delta_{N,\alpha}$ from a simultaneous Hoeffding band.
- A known second exponential-moment envelope gives an unbounded-outcome threshold from a simultaneous Chebyshev band.
- Under compatible marginals, each rule has rejection probability at most α, including singular points of the PSD cone.

HOW MANY WORLDS DOES A DECLARED TARGET REQUIRE?

Choose target regimes $S_1, \ldots, S_k$ and collect their nonempty intersections

$$\mathcal{J} = \left\{ \bigcap_{j \in L} S_j : \varnothing \neq L \subseteq [k] \right\}.$$

Theorem (Target-closure characterization)
The closure marginals construct a unique grouped-source joint law for the selected regimes exactly when the transformed grouped-source CGFs are admissible.

The closure contains at most $2^k - 1$ distinct worlds. Its size depends on the declared target collection.

WHICH THREE MARGINALS RECONSTRUCT A PAIR?

Let $I = S \cap T$. Compatibility requires $K_S - K_I$ and $K_T - K_I$ to be admissible. The pairwise reference has joint CGF

$$K^{\mathrm{F}}_{S,T}(u,v) = K_S(u) + K_T(v) - K_I(u) - K_I(v) + K_I(u+v).$$

The three marginals play separate roles.

- S and T supply the two endpoint laws.
- I supplies the shared source carried by both worlds.

WHAT IS THE FULL POLICY-GAIN DISTRIBUTION?

Substituting $(u, v) = (t, -t)$ gives

$$K^{\mathrm{F}}_{G_{S,T}}(t) = K_S(t) + K_T(-t) - K_I(t) - K_I(-t).$$

- For incomparable regimes, the marginals of S, T, and $S \cap T$ determine the gain law.
- Under $T \subseteq S$, the intersection equals T and $K_{G_{S,T}}(t) = K_S(t) - K_T(t)$.
- The ambient number of policy labels drops out of this pairwise formula.

WHICH MOMENTS REQUIRE THE INTERSECTION WORLD?

For every available order $r \geq 2$,

$$\kappa_r(G_{S,T}) = \begin{cases} \kappa_r(Y_S) - \kappa_r(Y_T), & r \text{ odd,} \\ \kappa_r(Y_S) + \kappa_r(Y_T) - 2\kappa_r(Y_I), & r \text{ even.} \end{cases}$$

- Endpoint marginals determine every odd gain cumulant.
- The intersection marginal completes every even order for incomparable regimes.
- The counts are identification minima over a factorized Poisson submodel.

WHY IS THE INTERSECTION INFORMATION NECESSARY?

For $0 \leq c \leq \min\{v_S, v_T\}$, take independent variables

$$U \sim N(0, c), \quad A \sim N(0, v_S - c), \quad B \sim N(0, v_T - c),$$

and set $Y_S = U + A$, $Y_T = U + B$.

- The endpoint laws remain $N(0, v_S)$ and $N(0, v_T)$ as c varies.
- The gain variance equals $v_S + v_T - 2c$.
- The shared variance $c = \mathrm{Var}(Y_{S \cap T})$ is the missing even-order information.

The Poisson construction in the paper extends the lower bound to every cumulant order and every observation design.

WHAT IS THE MINIMUM BUNDLE FOR MANY PAIRWISE GAINS?

For distinct regimes $S_1, \ldots, S_k$, define

$$\mathcal{E} = \{S_i\}_{i=1}^{k} \cup \{S_i \cap S_j\}_{i<j},$$

with duplicate worlds counted once.

Theorem (Minimum pairwise-gain bundle)
Fix an odd order $r_o \geq 3$ and an even order $r_e \geq 2$. Under cofactual orthogonality through order $\max\{r_o, r_e\}$, the marginals in $\mathcal{E}$ identify both declared cumulants of every pairwise gain. Every identifying marginal design contains $\mathcal{E}$.

Every positive additive world cost therefore selects the same unique bundle $\mathcal{E}$.

HOW DOES THE TARGET CHOOSE THE ACQUISITION PLAN?

Declared target	Marginal worlds acquired
One odd pairwise cumulant	S, T
One even nested cumulant	S, T
One even incomparable cumulant	$S, T, S \cap T$
All pairwise odd and even cumulants	distinct S_i and $S_i \cap S_j$

The row space of the observed incidence vectors determines sufficiency. A positive-intensity Poisson perturbation supplies the converse.

WHAT DO PAIRED OBSERVATIONS REVEAL?

Define the joint-CGF defect

$$\mathcal{D}_{S,T}(u,v) = K_{S,T}(u,v) - K^{\mathsf{F}}_{S,T}(u,v).$$

Then

$$K_{G_{S,T}}(t) = K^{\mathsf{F}}_{G_{S,T}}(t) + \mathcal{D}_{S,T}(t,-t).$$

- The anti-diagonal gives the exact error in the gain CGF.
- Mixed derivatives give every cumulant error.
- At order two, the error is $-2\{\mathrm{Cov}(Y_S, Y_T) - \mathrm{Var}(Y_I)\}$.

WHICH GAIN LAWS SURVIVE A GLOBAL SENSITIVITY BAND?

Assume the three closure marginals construct the grouped-source reference. Let $\mathfrak{C}_\rho$ contain continuous positive-definite functions Φ on $\mathbb{R}^2$ with

$$\Phi(u,0) = \varphi_S(u), \qquad \Phi(0,v) = \varphi_T(v),$$

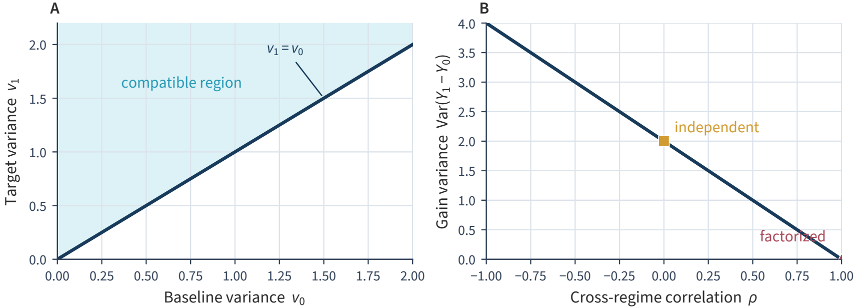
and

$$\left|\Phi(t,-t) - \Phi^{\mathsf{F}}_{S,T}(t,-t)\right| \le \rho(t) \quad \text{for every } t \in \mathbb{R}.$$

Theorem (Sharp global sensitivity)

The feasible gain laws are exactly those with $\varphi_G(t) = \Phi(t,-t)$ for some $\Phi \in \mathfrak{C}_\rho$. Every law in this set is attained by a coupling of the fixed endpoint marginals.

WHICH CONCLUSIONS USE MARGINALS AND WHICH USE PAIRED WORLDS?



Marginal compatibility, class-relative reconstruction, and paired specification restrictions answer different questions.

HOW DOES EACH OBSERVATION SCHEME ASSESS THE MODEL?

Data	Quantity	Assessment
Separate samples	transformed MGFs	marginal compatibility
Paired draws	$\mathcal{D}_{S,T}(u,v)$	joint specification
Paired moments	defect derivatives	order-specific restrictions
Adjusted pairs	conditional covariance	within-state restriction

The full functional test evaluates $\mathcal{D}_{S,T} \equiv 0$. The covariance Wald test evaluates its mixed second derivative at the origin.

WHERE CAN THE SHARED COMPONENTS COME FROM?

A fixed policy π and common transition kernel give the Bellman resolvent

$$V_S^\pi = (I - \beta P_\pi)^{-1} r_S^\pi.$$

If rewards decompose as $r_S = r_0 + \sum_{A \subseteq S, A \neq \varnothing} u_A$, then

$$V_S^\pi = V_0^\pi + \sum_{A \subseteq S, A \neq \varnothing} (I - \beta P_\pi)^{-1} u_A.$$

Independent reward modules remain independent after the common linear resolvent is applied component by component.

WHEN DOES REOPTIMIZATION PRESERVE THE SAME POLICY?

Let g be the minimum baseline action gap. Define

$$\varepsilon_S = \|r_S - r_0\|_\infty, \qquad \delta_S = \sup_{x,a} \sum_{x'} |P_S(x' \mid x, a) - P_0(x' \mid x, a)|,$$

$$\eta_S = \varepsilon_S + \beta \delta_S \|V_0^*\|_\infty.$$

Theorem (Policy stability)

If $\eta_S < (1-\beta)g/2$*, then* $\pi_S^* = \pi_0$.

The condition permits agents to solve the decision problem in each world. The action gap keeps the maximizing action unchanged inside a primitive neighborhood.

WHAT REMAINS WHEN THE POLICY CAN CHANGE?

For arbitrary perturbations,

$$\|V_S^* - V_S^{\pi_0}\|_\infty \le \frac{2\eta_S}{1-\beta}.$$

World-specific kernels add a resolvent term, yielding the paper's bound around the common-transition factorized reference. If $\eta_S(X) \le \bar{\eta}_S$ and

$\Pr\{g(X) \le z\} \le Cz^{\alpha}$, then

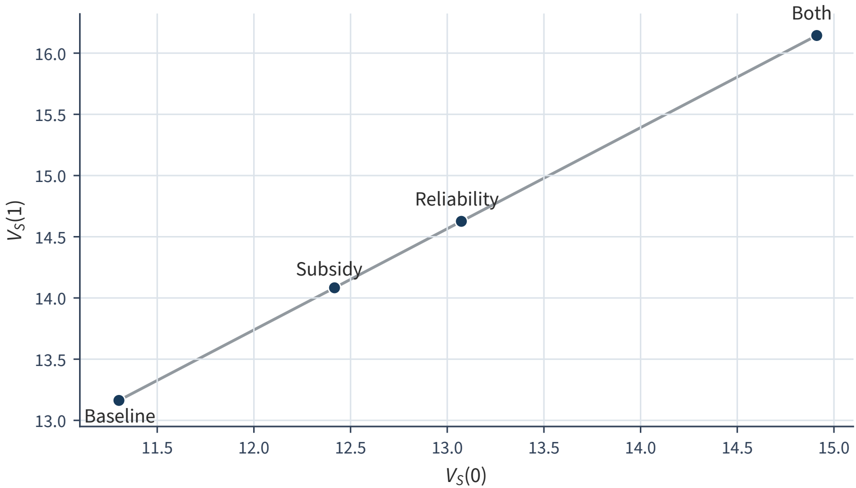
$$\Pr\{\pi_S^*(X) \ne \pi_0(X)\} \le C\left\{\frac{2\bar{\eta}_S}{1-\beta}\right\}^{\alpha}.$$

The same coupling argument bounds the total-variation distance between optimized and reference gain laws.

HOW DOES THE DYNAMIC EXAMPLE USE THE COORDINATES?

- The state records the current payoff status.
- Policy labels change reward and transition primitives in distinct combinations.
- Solving every world produces value vectors whose Möbius components record baseline, label, and interaction contributions.
- The action-gap calculation states when the same policy supports exact propagation and when the value bound supplies an approximation.

WHAT DOES THE DYNAMIC GEOMETRY LOOK LIKE?



The example's value vectors lie near the low-dimensional affine span generated by their cofactual components.

WHAT DOES THE FINITE-SAMPLE EXERCISE CHECK?

N	Component law	Robust Wald	Product shortcut
250	Independent	6.95	5.00
250	Orthogonal, dependent	6.60	25.25
1,000	Independent	4.70	4.75
1,000	Orthogonal, dependent	4.90	24.25

Rejection rates in percent from 2,000 replications at a 5 percent nominal level. The robust Wald statistic targets the covariance restriction. The product shortcut adds component independence and overrejects under dependent orthogonal components.

HOW DOES THE PAPER DIVIDE IMPORTED TOOLS AND TARGET RESULTS?

Imported tools

- Boolean Möbius inversion
- Exponential-convexity representation
- Bochner positive definiteness
- Bellman contraction bounds

Declared-target results

- Marginal-to-joint factorized reference
- Intersection-closure reconstruction
- Exact information minima and acquisition bundle
- Defect projection and attained gain-law sensitivity set

WHAT WORK DOES THE THEOREM SEQUENCE PERFORM?

1. Characterization states when the marginal system supports the coupling.
2. Construction recovers the joint reference law.
3. Minimality determines which worlds a declared target needs.
4. Specification and sensitivity quantify evidence against the reference.
5. Decision results connect the statistical structure to economic behavior.

WHAT SHOULD THE READER CARRY FORWARD?

1. Policy-gain distributions require a coupling across worlds.
2. Möbius-transformed marginals characterize one shared-component coupling class.
3. A declared target determines its own intersection closure and minimum marginal bundle.
4. Paired observations measure the exact gain-law departure, while positive definiteness gives the global feasible set.

WHICH DETAILS SUPPORT THE MAIN RESULTS?

Definitions, proof mechanisms, sampling conditions, and scope

WHICH DEFINITIONS ORGANIZE THE PAPER?

- A counterfactual system is the family $(Y_S)_{S \subseteq \mathcal{P}}$ on a common probability space.
- Cofactual components are the Boolean Möbius transform of that family.
- Cofactual factorization means mutual independence of all components.
- Local exponential integrability supplies law-determining CGFs near zero.
- Admissibility means that a transformed function is the CGF of a probability law.

WHAT IS THE ROW-SPACE IDENTIFICATION CRITERION?

Let $Z_{\mathcal{O}}$ collect incidence rows for observed worlds and let b define a world contrast. For order r,

$$q_r(b) = (Z^\top b)^{\circ r}.$$

The target cumulant is identified exactly when

$$q_r(b) \in \mathrm{row}(Z_{\mathcal{O}}).$$

Every solution of $Z_{\mathcal{O}}^\top \lambda = q_r(b)$ gives

$$\kappa_r(b^\top Y) = \lambda^\top k_{r,\mathcal{O}}.$$

HOW DOES THE ALL-ORDER NECESSITY PROOF WORK?

1. Give every independent component a strictly positive Poisson intensity.
2. Perturb the intensity vector in the null space of $Z_{\mathcal{O}}$.
3. The complete observed marginal laws stay fixed because their Poisson means stay fixed.
4. A target loading outside the observed row space changes under a suitable perturbation.

For pairwise gains, component loadings lie in $\{-1, 0, 1\}$. Their odd powers equal the loading and their positive even powers equal its square. This parity relation yields the all-order necessity argument.

WHY IS THE MULTI-PAIR BUNDLE UNIQUELY MINIMAL?

Odd and even pairwise loadings generate differences between endpoint incidence rows and their intersection rows.

- These differences form the edge space of a connected graph on $\mathcal{E}$.
- All Boolean incidence rows form a basis.
- A span containing an edge difference contains both basis rows on that edge.
- Connectivity therefore requires every world in $\mathcal{E}$.

Positive additive costs make every additional world strictly costly, giving the unique minimum-cost result.

WHY IS THE GLOBAL SENSITIVITY SET SHARP?

1. Every coupling supplies a continuous normalized positive-definite Φ with the declared marginal axes.
2. Its gain characteristic function is the anti-diagonal $\Phi(t, -t)$.
3. Conversely, Bochner's theorem maps each feasible Φ to a probability measure on $\mathbb{R}^2$.
4. The axis restrictions fix its marginals, and the gain pushforward attains the projected law.

Finite grids give PSD completion restrictions. The infinite hierarchy gives the population set.

WHICH SAMPLING CONDITIONS SUPPORT COMPATIBILITY INFERENCE?

- The expanding-grid sieve uses uniform root-N estimation of the world CGFs on a compact interval.
- The bounded-support calibration propagates a simultaneous Hoeffding band through log, Möbius inversion, exponentiation, and Weyl's inequality.
- The unbounded calibration uses a known second exponential-moment envelope and the same deterministic propagation.
- The finite-sample statements control rejection throughout the compatible class.

HOW DO THE FORMAL RESULTS DIVIDE THE ARGUMENT?

Role	Main result
Existence	Marginal-to-joint characterization
Target sufficiency	Target-closure and row-space theorems
Target necessity	Poisson separation and bundle minimality
Gain-law sensitivity	Defect identity and global sensitivity
Behavioral stability	Policy stability and margin bounds
Specification	PSD and paired-moment procedures

WHICH SCOPE CONDITIONS GOVERN THE CLAIMS?

- Full-law reconstruction uses local exponential integrability and admissible transformed CGFs.
- Joint-law uniqueness is relative to the factorized class selected by the component independence restriction.
- Cumulant results require the stated moment order and the corresponding cofactual orthogonality condition.
- Global sensitivity uses characteristic functions on $\mathbb{R}^2$ and fixed endpoint marginals.
- Bellman exactness uses the action-gap region; the value and margin bounds cover departures outside that region.